Solutions to Quiz 2, 2019

Problem 1

Use the Bio-Savart law,

$$d\mathbf{H} = \frac{Id\mathbf{l}\times\mathbf{R}}{4\pi R^3}, \tag{1}$$

1) Consider the horizontal arms first. On those arms, $Id\mathbf{l} = Idx\,\mathbf{a}_x$ and $\mathbf{R} = \pm x\mathbf{a}_x$. It follows at once from Eq. 1 that since $Id\mathbf{l}\times\mathbf{R} = 0$, we have no contribution to the field at the center due to the horizontal arms.

2) Consider now the semi-circle. On the semi-circle, $Id\mathbf{l} = -Ibd\phi\,\mathbf{a}_\phi$. The minus sign comes from the fact that the positive direction of $\mathbf{a}_\phi$ is counterclockwise. Next, $\mathbf{R} = b\mathbf{a}_\rho$ and $R = b$. It then follows from Eq.1 that

$$d\mathbf{H} = \frac{-Ibd\phi(\mathbf{a}_\phi\times\mathbf{a}_\rho)b}{4\pi b^3} = \frac{Id\phi}{4\pi b}\mathbf{a}_z. \tag{2}$$

Thus,

$$\mathbf{H} = \int_\pi^0 d\phi\,\frac{I}{4\pi b}\mathbf{a}_z = -\frac{I}{4b}\mathbf{a}_z. \tag{3}$$

The limits of integration correspond to going from the left to the right end of the semi-circle. The minus sign in Eq. 2 implies that the field is directed into the plane of the loop.

Problem 2

1) Method I. Working in the spherical coordinates,

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix}\mathbf{a}_r & r\mathbf{a}_\theta & r\sin\theta\mathbf{a}_\phi\\ \partial_r & \partial_\theta & \partial_\phi\\ 0 & 0 & \frac{1}{2}r^2\sin^2\theta\end{vmatrix} = \mathbf{a}_r\cos\theta - \mathbf{a}_\theta\sin\theta.$$

2) Method II. Alternatively, observing that $\mathbf{A} = \frac{1}{2}\rho\mathbf{a}_\phi$, we can work in the cylindrical coordinates,

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{1}{\rho}\begin{vmatrix}\mathbf{a}_\rho & \rho\mathbf{a}_\phi & \mathbf{a}_z\\ \partial_\rho & \partial_\phi & \partial_z\\ 0 & \frac{1}{2}\rho^2 & 0\end{vmatrix} = \mathbf{a}_z.$$

The two answers are equivalent as can be readily inferred from your formula sheet.

Problem 3

a) By definition, $I = \int d\mathbf{S} \cdot \mathbf{J}$. In this case, $d\mathbf{S} = Rd\phi dz\, \mathbf{a}_\rho$ on the cylindrical surface. It follows that

$$I = \int_0^L dz \int_0^{2\pi} d\phi\, (\mathbf{a}_\rho \cdot \mathbf{a}_\rho)\, RJ_0(R/R) = 2\pi RLJ_0.$$

b) The divergence of the current density reads,

$$\nabla \cdot \mathbf{J} = \tfrac{1}{\rho}\partial_\rho(\rho J_\rho) = \tfrac{1}{\rho}\partial_\rho\left(J_0\tfrac{\rho^2}{R}\right) = 2J_0/R. \tag{4}$$

The continuity equation implies that

$$\partial_t \rho_v = -\nabla \cdot \mathbf{J} = -2J_0/R. \tag{5}$$

It follows by integrating Eq. 5 that

$$\rho_v(\mathbf{r}, t) = \rho_v(\mathbf{r}, 0) - 2J_0 t/R = \rho_{v0} - 2J_0 t/R. \tag{6}$$

c) It follows at once from Eq. 6 that the current density vanishes at the finite time

$$t_* = \frac{\rho_{v0} R}{2J_0}.$$